

# An analogue of the Erdős-Ko-Rado theorem for multisets

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## Abstract

## 1 Introduction

### 1.1 Definitions, Notation

Let  $n$  and  $l$  be positive integers and let  $M(n, l) = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq l\}$  be an  $n \times l$  rectangle. We call  $A \subseteq M(n, l)$  a  $k$ -multiset if the cardinality of  $A$  is  $k$  and  $(i, j) \in A$  implies  $(i, j') \in A$  for all  $j' \leq j$ . We think of multisets as sets with multiplicities, but it helps finding short and precise notation if we identify them with these special subsets of the rectangle. We denote the multiplicity of  $i$  in  $F$  by  $m(i, F)$ , i.e.  $m(i, F) = \max\{s : (i, s) \in F\}$  ( $m(i, F) \leq l$  by definition).

Let  $\mathcal{F}$  be a family of  $k$ -multisets of  $M(n, l)$ . We call  $\mathcal{F}$   $t$ -intersecting if  $|F_1 \cap F_2| \geq t$  for all  $F_1, F_2 \in \mathcal{F}$ . Let  $\mathcal{M}(n, l, k, t) = \{\mathcal{F} : \mathcal{F} \text{ is } t\text{-intersecting set of } k\text{-multisets of } M(n, l)\}$ , i.e. the class of  $t$ -intersecting families of  $k$ -multisets.

Let  $\mathcal{F} \in \mathcal{M}(n, l, k, t)$ . We call  $T \subseteq M(n, l)$  a  $t$ -kernel for  $\mathcal{F}$  if  $|F_1 \cap F_2 \cap T| \geq t$  for all  $F_1, F_2 \in \mathcal{F}$ .

### 1.2 History

Let us briefly summarize some results using our notation.

**Theorem 1.1.** (Erdős, Ko, Rado, [3]) *If  $n \geq 2k$  and  $\mathcal{F} \in \mathcal{M}(n, 1, k, 1)$  then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$

*If  $n > 2k$ , then equality holds if and only if all members of  $\mathcal{F}$  contain a fixed element of  $[n]$ .*

They also proved that if  $n$  is large enough, every member of the largest  $t$ -intersecting family of sets contains a fixed  $t$ -element set, but did not give the optimal threshold. Frankl [6] showed for  $t \geq 15$  and Wilson [7] for every  $t$  that the optimal threshold is  $n = (k - t + 1)(t + 1)$ . Finally, Ahlswede and Khachatrian [1] determined the maximum families for all values of  $n$ .

**Theorem 1.2.** (Ahlsweide, Khachatrian [1])

Let  $t \leq k \leq n$  and  $\mathcal{A}_{n,k,t,i} = \{A : A \subseteq [n], |A| = k, |A \cap [t+2i]| \geq t+i\}$ . If  $\mathcal{F} \in \mathcal{M}(n, 1, k, t)$ , then

$$|\mathcal{F}| \leq \max_i |\mathcal{A}_{n,k,t,i}| = AK(n, k, t).$$

**Theorem 1.3.** (Meagher, Purdy [5]) If  $n \geq k+1$  and  $\mathcal{F} \in \mathcal{M}(n, k, k, 1)$ , then

$$|\mathcal{F}| \leq \binom{n+k-2}{k-1}.$$

If  $n > k+1$ , then equality holds if and only if all members of  $\mathcal{F}$  contain a fixed element of  $M(n, k)$ .

### 1.3 Conjectures

Brockman and Kay stated the following conjecture [2]:

**Conjecture 1.4.** ([2], Conjecture 5.2.)

There is  $n_0(k, t)$  such that if  $n \geq n_0(k, t)$  and  $\mathcal{F} \in \mathcal{M}(n, k, k, t)$ , then

$$|\mathcal{F}| \leq \binom{n+k-t-1}{k-t}.$$

Furthermore, equality is achieved if and only if each member of  $\mathcal{F}$  contains a fixed  $t$ -multiset of  $M(n, k)$ .

Meagher and Purdy also gave a possible candidate for the threshold  $n_0(k, t)$ .

**Conjecture 1.5.** ([5], Conjecture 4.1.)

Let  $k, n$  and  $t$  be positive integers with  $t \leq k$ ,  $t(k-t) + 2 \leq n$  and  $\mathcal{F} \in \mathcal{M}(n, k, k, t)$ , then

$$|\mathcal{F}| \leq \binom{n+k-t-1}{k-t}.$$

Moreover, if  $n > t(k-t) + 2$ , then equality holds if and only if all members of  $\mathcal{F}$  contain a fixed  $t$ -multiset of  $M(n, k)$ .

Note that if  $n < t(k-t) + 2$ , then the family consisting of all multisets which contain a fixed  $t$ -multiset of  $M(n, k)$  still has cardinality  $\binom{n+k-t-1}{k-t}$ , but cannot be the largest. Indeed, if we fix a  $t+2$ -multiset  $T$  and consider the family of the multisets  $F$  with  $|F \cap T| \geq t+1$ , we get a larger family.

## 1.4 Results

The main idea of our proof is the following: instead of the well-known *left-compression* operation, which is a usual method in the theory of intersecting families, we define (in two different ways) an operation on  $\mathcal{M}(n, l, k, t)$  which can be called a kind of *down-compression*.

**Theorem 1.6.** *Let  $1 \leq t \leq k$ ,  $2k - t \leq n$  and  $l$  be arbitrary. There exists*

$$f : \mathcal{M}(n, l, k, t) \rightarrow \mathcal{M}(n, l, k, t)$$

*satisfying the following properties:*

- (i)  $|\mathcal{F}| = |f(\mathcal{F})|$  for all  $\mathcal{F} \in \mathcal{M}(n, l, k, t)$ .
- (ii)  $M(n, 1)$  is a  $t$ -kernel for  $f(\mathcal{F})$ .

Using Theorem 1.6 we prove the following theorem which not only verifies Conjecture 1.5, but also gives the maximum cardinality of  $t$ -intersecting families of multisets in the case  $2k - t \leq n \leq t(k - t) + 2$ .

**Theorem 1.7.** *Let  $1 \leq t \leq k$  and  $2k - t \leq n$ . If  $\mathcal{F} \in \mathcal{M}(n, k, k, t)$  then*

$$|\mathcal{F}| \leq AK(n + k - 1, k, t).$$

## 1.5 Warm up before the proofs

Let us remark that one can relatively easily verify Conjecture 1.4. Let  $T$  be a  $t$ -multiset. For any family  $\mathcal{F}$  let  $\mathcal{F}_T = \{F \in \mathcal{F} : T \subseteq F\}$ .

**Lemma 1.8.** *Let  $\mathcal{F}$  be a  $t$ -intersecting family of multisets and  $T$  be an arbitrary  $t$ -multiset. Then either  $\mathcal{F}_T = \mathcal{F}$  or  $|\mathcal{F}_T| = O_n(n^{k-t-1})$ .*

*Proof.* If  $\mathcal{F}_T \neq \mathcal{F}$ , then there is a multiset  $F \in \mathcal{F}$  which does not contain  $T$ , hence  $|F \cap T| \leq t - 1$ . Every member of  $\mathcal{F}_T$  intersects  $F$  in these at most  $t - 1$  elements and at least one other element. Hence they contain  $T$ , one element of  $F \setminus T$ , and at most  $k - t - 1$  other elements. The element of  $F \setminus T$  can be chosen less than  $k$  ways, and the other  $k - t - 1$  elements have to be chosen out of the  $nl$  elements of the rectangle  $M(n, l)$ . There are  $O_n(n^{k-t-1})$  ways to do that. □

**Corollary 1.9** (Conjecture 1.4). *There is  $n_0(k, t)$  such that if  $n \geq n_0(k, t)$  and  $\mathcal{F} \in \mathcal{M}(n, k, k, t)$ , then*

$$|\mathcal{F}| \leq \binom{n + k - t - 1}{k - t}.$$

*Furthermore, equality is achieved if and only if each member of  $\mathcal{F}$  contains fixed  $t$  elements.*

*Proof.* Let  $\mathcal{F} \in \mathcal{M}(n, k, k, t)$  of maximum cardinality. If  $\mathcal{F}_T = \mathcal{F}$  for a  $t$ -multiset  $T$ , the statement follows. If not, then let us fix an  $F \in \mathcal{F}$ . Every member of  $\mathcal{F}$  contains a  $t$ -multiset which is also contained in  $F$ , hence  $\bigcup \{\mathcal{F}_T : T \subset F, |T| = t\} = \mathcal{F}$ . Thus  $|\mathcal{F}| \leq \sum_{T \subset F, |T|=t} |\mathcal{F}_T|$ . By Lemma 1.8  $|\mathcal{F}_T| = O_n(n^{k-t-1})$ , and there are  $\binom{k}{t}$  members of the sum, hence  $|\mathcal{F}| \leq \binom{k}{t} O_n(n^{k-t-1}) < \binom{n+k-t-1}{k-t}$  if  $n$  is large enough.  $\square$

To attack Conjecture 1.5 at first we developed a straight-forward generalization of shifting. However, we could not give a threshold lower than  $\Omega(kt \log k)$  using this method. Still we believe it is worth mentioning, as it might be useful solving other related problems.

For  $F \subseteq M(n, l)$   $k$ -multiset and  $i < j$  let us suppose  $m(j, F) - m(i, F) > 0$ . Let  $F'$  be the result if we exchange column  $j$  and column  $i$ , i.e.  $F' = F \setminus \{(j, m(i, F) + 1), \dots, (j, m(j, F))\} \cup \{(i, m(i, F) + 1), \dots, (i, m(j, F))\}$ . Let  $\mathcal{F} \in \mathcal{M}(n, l, k, t)$  and  $F \in \mathcal{F}$ . Then

$$c_{i,j}(F) = \begin{cases} F' & \text{if } m(F, j) - m(F, i) > 0 \text{ and } F' \notin \mathcal{F} \\ F & \text{otherwise.} \end{cases}$$

Let us use the following notation:  $c_{i,j}(\mathcal{F}) = \{c_{i,j}(F) : F \in \mathcal{F}\}$ . Note that this is the same as the well-known shifting operation in case  $\mathcal{F} \in \mathcal{M}(n, 1, k, t)$ .

**Lemma 1.10.**  $c_{i,j}(\mathcal{F}) \in \mathcal{M}(n, k, k, t)$  ( $i < j$ ) for  $\mathcal{F} \in \mathcal{M}(n, k, k, t)$ .

*Proof.* By contradiction suppose there are  $F_1, F_2 \in \mathcal{F}$  with  $|c_{i,j}(F_1) \cap c_{i,j}(F_2)| < t$ .

If both or neither of  $c_{i,j}(F_1)$  and  $c_{i,j}(F_2)$  are members of  $\mathcal{F}$ , their intersection obviously has size at least  $t$ . Hence wlog we can assume  $c_{i,j}(F_1) = F'_1$  and  $c_{i,j}(F_2) = F_2$ .

Let  $x$  be the cardinality of the intersection of  $F_1$  and  $F_2$  in the complement of the union of the  $i$ th and  $j$ th column.

*Case 1:*  $m(F_2, i) \geq m(F_2, j)$ . We know that  $m(F_1, i) < m(F_1, j)$ ,

$$\begin{aligned} x + \min\{m(F_2, i), m(F_1, i)\} + \min\{m(F_2, j), m(F_1, j)\} &\geq t \quad \text{and} \\ x + \min\{m(F_2, i), m(F_1, j)\} + \min\{m(F_2, j), m(F_1, i)\} &< t. \end{aligned}$$

If  $m(F_1, j)$  is the largest of the four numbers, then  $\min\{m(F_2, i), m(F_1, i)\} + m(F_2, j) > m(F_2, i) + \min\{m(F_2, j), m(F_1, i)\}$ , but here the left hand side is at least  $m(F_2, j) + m(F_2, i)$ , the right hand side can be smaller only if  $m(F_1, i) < m(F_2, j)$ , but in that case one can easily see that the left hand side is even smaller. If not  $m(F_1, j)$ , then  $m(F_2, i)$  is (one of) the largest of the four numbers, and the proof goes similarly.

*Case 2:*  $m(F_2, i) < m(F_2, j)$  but  $F'_2 \in \mathcal{F}$ . We know that  $|F'_1 \cap F_2| = |F'_2 \cap F_1| \geq t$ , a contradiction.  $\square$

**Remark.** It's worth mentioning that there is an even more straightforward generalization of shifting, when we just decrease the multiplicity in column  $j$  by one and increase it in column  $i$  by one. Let  $F' = F \cup \{(i, m(i, F)) + 1\} \setminus \{(j, m(j, F))\}$ , and

$$c'_{i,j}(F) = \begin{cases} F' & \text{if } m(j, F) > m(i, F) \text{ and } F' \notin \mathcal{F} \\ F & \text{otherwise.} \end{cases}$$

But this operation does not preserve the  $t$ -intersecting property. However, if we apply our shifting operation to a maximum  $t$ -intersecting family and for every pair  $(i, j)$ , the resulting family will be also shifted according to this second kind of shifting, meaning that applying this second operation does not change the family.

After applying  $c_{i,j}$  for every pair  $i, j$ , the resulting shifted family has several different  $t$ -kernels. For example the union of two rectangles  $t^{1/2} \times (2k - t) \cup t \times \frac{2k}{t^{1/2}}$  is a  $t$ -kernel, or the set where the members  $(x, y)$  satisfy  $yx \leq k, x \leq 2k - t, y \leq k, x, y \geq 0$ .

But using these  $t$ -kernels and some algebra, we failed to go through the  $O(kt \log k) \leq n$  barrier. That's the reason we had to develop the *down compressing* techniques described in the next section.

## 2 Proof of Theorem 1.6

### 2.1 First proof - a constructive one

#### 2.1.1 A lemma about interval systems

We will consider multisets which contain almost exactly the same elements, they differ only in two columns. More precisely, we are interested in multisets whose symmetric difference is a subset of  $\{(i, j) : 1 \leq j \leq l\} \cup \{(i', j) : 1 \leq j \leq l\}$  with  $i \neq i'$ . If we are given two such multisets, we can consider the two columns together, one going from  $l$  to 1 and the other going from 1 to  $l$ . This way the two multisets form a subinterval of an interval of length  $2l$  (where interval means set of consecutive integers). Hence we examine families of intervals.

Let  $X = \{1, \dots, 2l\}$ . Let  $Y \subset X$  be an interval and  $p$  be an integer with  $p \leq |Y|$ , then we define a family of intervals  $I(p, Y)$  to be all the  $p$ -element subintervals of  $Y$ . Let  $\mathcal{I}$  denote the class of these families.

We consider a shifted version, where the intervals are pushed to the middle. Let  $\varphi(Y) = \{l - \lfloor |Y|/2 \rfloor, \dots, l + \lceil |Y|/2 \rceil - 1\}$  and  $\varphi(I(p, Y)) = I(p, \varphi(Y))$ . We will show that this operation does not decrease the size of the intersection of two families in  $\mathcal{I}$ . Let  $d(I(p, Y), I(q, Y')) = \min\{|I \cap J| : I \in I(p, Y), J \in I(q, Y')\}$ .

**Lemma 2.1.**  $d(I(p, \varphi(Y)), I(q, \varphi(Y'))) \geq d(I(p, Y), I(q, Y'))$  for all possible  $p, q, Y$  and  $Y'$ .

*Proof.* Obviously the smallest intersection is the intersection of the first interval in one of the families and the last interval in the other family. As the length of the intervals are always  $p$  and  $q$ , the only thing that matters is the difference between the starting and ending points of  $\varphi(Y)$  and  $\varphi(Y')$ . More precisely we want to minimize the largest of  $y = \max \varphi(Y) - \min \varphi(Y')$  and  $y' = \max \varphi(Y') - \min \varphi(Y)$ . As  $\max \varphi(Y) - \min \varphi(Y) = |Y| - 1$  and  $\max \varphi(Y') - \min \varphi(Y') = |Y'| - 1$ , we know that  $y + y'$  is constant, hence we get the minimum if  $y$  and  $y'$  is as close as possible. One can easily see that our shifted system gives this. □

### 2.1.2 Interval systems and families of multisets

Now to apply the method of the previous subsection, we fix  $n - 2$  coordinates, i.e. we are given  $i < j \leq n$  and  $g : ([n] \setminus \{i, j\}) \rightarrow [1, l]$ . Let  $\mathcal{F}_g = \{F \in \mathcal{F} : m(r, F) = g(r) \text{ for every } r \neq i, j\}$ . It implies  $m(i, F) + m(j, F)$  is the same number  $s = s(g)$  for every member  $F \in \mathcal{F}_g$ .

Let us consider now the case  $\mathcal{F}$  is maximal, i.e. no  $k$ -multiset can be added to it without violating the  $t$ -intersecting property. We show that it implies that the integers  $m(i, F)$  are consecutive for  $F \in \mathcal{F}_g$ . Let  $m_i = \min\{m(i, F) : F \in \mathcal{F}_g\}$  and  $M_i = \max\{m(i, F) : F \in \mathcal{F}_g\}$ . We define  $m_j$  and  $M_j$  similarly. Let us consider a set  $F \notin \mathcal{F}$  which satisfies  $m(r, F) = g(r)$  for all  $r \neq i, j$  and also  $m_i \leq m(i, F) \leq M_i$ , and consequently  $m_j \leq m(j, F) \leq M_j$ . It is easy to see that  $F$  can be added to  $\mathcal{F}$  without violating the  $t$ -intersecting property (and then it belongs to  $\mathcal{F}_g$ ).

Now we give a bijection between these type of families and interval systems. We lay down both columns, such that column  $i$  starts at its top, and column  $j$  start at its bottom. Then move them next to each other to form an interval. More precisely let  $\Psi((i, u)) = l - u + 1$  and  $\Psi((j, u)) = l + u$ . For a multiset  $F$  let  $\Psi(F) = \{\Psi((i, u)) : (i, u) \in F\} \cup \{\Psi((j, u)) : (j, u) \in F\}$  and for a family of multisets  $\mathcal{F}$  let  $\Psi(\mathcal{F}) = \{\Psi(F) : F \in \mathcal{F}\}$ .

We show that  $\Psi(\mathcal{F}_g) \in \mathcal{I}$ . It is obvious that  $\Psi(F)$  is an interval for any multiset  $F$ , and that the length of those intervals is the same number (more precisely  $s$ ) for every member  $F \in \mathcal{F}_g$ . We need to show that the intervals  $\Psi(F)$  (where  $F \in \mathcal{F}_g$ ) are all the subintervals of an interval  $Y$ . It is enough to show that the starting points of these intervals are consecutive integers. The starting points of the intervals  $\Psi(F)$  are  $\Psi((i, m(i, F)))$ , and it is easy to see that they are consecutive if and only if  $m(i, F)$  are consecutive.

Since  $\Psi$  is a bijection, an interval system also defines a family in the two columns  $i$  and  $j$ . Let us examine what family we get after applying operation  $\varphi$  from the previous section, i.e. what  $\mathcal{F}' = \Psi^{-1}(\varphi(\Psi(\mathcal{F}_g)))$  is. Obviously it is a family of  $s$ -multisets with the same cardinality as  $\mathcal{F}_g$ . Simple calculations show that they are the  $s$ -multisets with  $m(i, F) \leq \lfloor (m_i + M_j)/2 \rfloor + 1$  and  $m(j, F) \leq \lceil (m_i + M_j)/2 \rceil - 1$ .

### 2.1.3 The construction of $f$

Let  $\psi(\mathcal{F}_g) = \Psi^{-1}(\varphi(\Psi(\mathcal{F}_g)))$ , i.e. the family we get from  $\mathcal{F}_g$  by keeping everything in the other  $n - 2$  columns, but making it balanced in the columns  $i$  and  $j$  in the following sense. It contains all the  $k$ -multisets where  $m(i, F) \leq \lfloor (m_i + M_j)/2 \rfloor + 1$ ,  $m(j, F) \leq \lceil (m_i + M_j)/2 \rceil - 1$  and the other coordinates are given by  $g$ .

Now let us recall that  $i$  and  $j$  are fixed. Let  $G_{i,j}$  be the set of every  $g : ([n] \setminus \{i, j\}) \rightarrow [1, l]$ , i.e. every possible way to fix the other  $n - 2$  coordinates. Clearly  $\mathcal{F} = \cup\{\mathcal{F}_g : g \in G_{i,j}\}$  and they are all disjoint. Let  $\psi_{i,j}(\mathcal{F})$  denote the result of applying the appropriate  $\psi$  operation for every  $g$  at the same time, i.e.  $\psi_{i,j}(\mathcal{F}) = \cup\{\psi(\mathcal{F}_g) : g \in G_{i,j}\}$ .

**Lemma 2.2.** *If  $\mathcal{F}$  is  $t$ -intersecting, then  $\psi_{i,j}(\mathcal{F})$  is  $t$ -intersecting.*

*Proof.* Suppose there are  $F_1, F_2 \in \psi_{i,j}(\mathcal{F})$  with  $|F_1 \cap F_2| < t$ . Let  $F_1 \in \psi_{i,j}(\mathcal{F}_{g_1})$  and  $F_2 \in \psi_{i,j}(\mathcal{F}_{g_2})$ . Let  $\Psi(\mathcal{F}_{g_1}) = I(p_1, Y_1)$  and  $\Psi(\mathcal{F}_{g_2}) = I(p_2, Y_2)$ . Then  $\Psi(\psi_{i,j}(\mathcal{F}_{g_1})) = \varphi(I(p_1, Y_1))$  and

$\Psi(\psi_{i,j}(\mathcal{F}_{g_2})) = \varphi(I(p_2, Y_2))$ . It is important to see that  $\Psi$  is defined on the elements of  $M(n, l)$  such a way that the size of the intersection is the same after applying  $\Psi$ .

By Lemma 2.1  $d(I(p_1, \varphi(Y_1)), I(p_2, \varphi(Y_2))) \geq d(I(p_1, Y_1), I(p_2, Y_2))$ , which means there is a member of  $\mathcal{F}_{g_1}$  and a member of  $\mathcal{F}_{g_2}$  such that their intersection has size at most the size of the smallest intersection between members of  $\psi_{i,j}(\mathcal{F}_{g_1})$ , which is less than  $t$ , a contradiction.

Note that it does not matter if  $g_1$  is equal to  $g_2$ . □

**Lemma 2.3.** *If  $\psi_{i,j}(\mathcal{F}) \neq \mathcal{F}$  then*

$$\sum_{F' \in \psi_{i,j}(\mathcal{F})} \left[ |\mathcal{F}|nk^2 \sum_{i \in [n]} (m(i, F'))^2 + \sum_{i \in [n]} i(m(i, F')) \right] < \sum_{F \in \mathcal{F}} \left[ |\mathcal{F}|nk^2 \sum_{i \in [n]} (m(i, F))^2 + \sum_{i \in [n]} i(m(i, F)) \right] \quad (1)$$

*Proof.* Trivial by the symmetrization. □

Now we are ready to define  $f(\mathcal{F})$ . If there is a pair  $(i, j)$  such that  $\psi_{i,j}(\mathcal{F}) \neq \mathcal{F}$ , let us replace  $\mathcal{F}$  by  $\psi_{i,j}(\mathcal{F})$ , and repeat this step. Lemma 2.3 implies that it can be done only finitely many times, after that we arrive to a family  $\mathcal{F}'$  such that  $\psi_{i,j}(\mathcal{F}') = \mathcal{F}'$  for every pair  $(i, j)$ . This family is denoted by  $f(\mathcal{F})$ .

We would like to prove that  $f$  satisfies Theorem 1.6 (ii):

**Lemma 2.4.**  $|F_1 \cap F_2 \cap M(n, 1)| \geq t$  for all  $F_1, F_2 \in f(\mathcal{F})$ .

*Proof.* We argue by contradiction. Let us choose  $F_1$  and  $F_2$  such a way that  $|F_1 \cap F_2 \cap M(n, 1)|$  is the smallest (definitely less than  $t$ ), and among those  $|F_1 \cap F_2|$  is the smallest (definitely at least  $t$ ). Then there is a coordinate where both  $F_1$  and  $F_2$  have at least 2, and this implies there is an other coordinate, where both have 0, as  $2k - t \leq n$ . More precisely, there is an  $i \leq n$  with  $2 \leq \min\{m(i, F_1), m(i, F_2)\}$  and a  $j \leq n$  with  $m(j, F_1) = m(j, F_2) = 0$ . Let  $F'_1$  be defined the following way:  $m(j, F'_1) = 1, m(i, F'_1) = m(i, F_1) - 1$  and  $m(s, F'_1) = m(s, F_1)$  for  $s \leq n, s \neq i, j$ . One can easily see that  $F'_1 \in \psi_{i,j}(f(\mathcal{F})) = f(\mathcal{F})$ . However,  $|F'_1 \cap F_2| < |F_1 \cap F_2|$  and  $|F'_1 \cap F_2 \cap M(n, 1)| = |F_1 \cap F_2 \cap M(n, 1)|$ , a contradiction. □

To finish the proof of Theorem 1.6 we have to deal with the case  $\mathcal{F}$  is not maximal (even though it is not needed in order to prove Theorem 1.7). For sake of brevity here we just give a sketch.

Note that  $\Psi$  can be similarly defined in this case. The main difference is that the resulting family of intervals is not in  $\mathcal{I}$ , as it does not contain all the subintervals of an interval. Also note that  $\varphi(I(p, Y))$  is determined by the number and length of the intervals in  $I(p, Y)$ . Using this we can extend the definition of  $\varphi$  to any family of intervals. This way we can define  $\psi_{i,j}$  as well. What happens is that besides being more balanced in the columns  $i$  and  $j$ , the multisets in  $\mathcal{F}_g$  are also pushed closer to each other. Hence one can easily see that the intersections cannot be smaller in this case, which finishes the proof.

## 2.2 Second proof - a less constructive one

*Proof of Theorem 1.6:*

For  $F \in \mathcal{F} \in \mathcal{M}(n, l, k, t)$ ,  $i \leq n$ ,  $s \leq m(i, F)$  and  $j \leq n$  let

$$F' = F \setminus \bigcup_{s \leq t \leq m(i, F)} (i, t) \bigcup (\bigcup_{1 \leq l \leq m(i, F) - s + 1} (j, l)).$$

Using this notation we define another shifting operation.

**Definition 2.5.**  $S(i, s)(j, 1)(F) = \begin{cases} F' & \text{if } (j, 1) \notin F \text{ and } F' \notin \mathcal{F} \\ F & \text{otherwise.} \end{cases}$

For  $\mathcal{F} \in \mathcal{M}(n, l, k, t)$  let  $S(i, s)(j, 1)(\mathcal{F}) = \{S(i, s)(j, 1)(F) : F \in \mathcal{F}\}$ .

For  $\mathcal{F} \in \mathcal{M}(n, l, k, t)$  let  $\mathcal{K}(\mathcal{F})$  be the set of  $t$ -kernels of  $\mathcal{F}$  which contain  $M(n, 1)$  and also multisets. For  $T \in \mathcal{K}(\mathcal{F})$  let  $T_{>1} = T \setminus M(n, 1)$ . We would like to define an operation on  $\mathcal{M}(n, l, k, t)$  which decreases  $\min\{|T_{>1}| : T \in \mathcal{K}(\mathcal{F})\}$  for any  $\mathcal{F} \in \mathcal{M}(n, l, k, t)$  in case it is positive.

Let us apply  $S(i, m(i, T))(1, 1)$  to  $\mathcal{F}$ , then  $S(i, m(i, T))(2, 1)$  to the resulting family, and so on. Let  $\mathcal{F}'$  be the resulting family after applying  $S(i, m(i, T))(n, 1)$ , i.e

$$\mathcal{F}' = S(i, m(i, T))(n, 1)[\dots[S(i, m(i, T))(2, 1)[S(i, m(i, T))(1, 1)(\mathcal{F})]]\dots].$$

**Lemma 2.6.** *Let  $\mathcal{F} \in \mathcal{M}(n, l, k, t)$ ,  $T \in \mathcal{K}(\mathcal{F})$  satisfying  $|T_{>1}| > 0$  and let  $1 \leq i \leq n$ ,  $2 \leq m(i, T)$ . Then:*

- (i)  $\mathcal{F}' \in \mathcal{M}(n, l, k, t)$  and  $|\mathcal{F}| = |\mathcal{F}'|$ ,
- (ii)  $(T \setminus (i, m(i, T))) \in \mathcal{K}(\mathcal{F}')$ .

*Proof.*

- proof of (i):

The facts that  $S(i, m(i, T))(n, 1)[\dots[S(i, m(i, T))(2, 1)[S(i, m(i, T))(1, 1)(F)]]\dots] \subseteq M(n, k)$  with cardinality  $k$  for any  $F \in \mathcal{F}$  and that  $|\mathcal{F}'| = |\mathcal{F}|$ , are trivial.

**Claim 2.7.**  $\mathcal{F}'$  is  $t$ -intersecting.

*Proof.* It is enough to prove that  $S(i, m(i, T))(1, 1)(\mathcal{F})$  is  $t$ -intersecting and that  $T$  is a  $t$ -kernel for the new family, i.e.  $T \in \mathcal{K}(S(i, m(i, T))(1, 1)(\mathcal{F}))$ , since repeatedly applying this fact we will get the claim.

Choose two arbitrary members. As usual, it is easy to handle the cases when both or neither is a member of the original family  $\mathcal{F}$ . Hence we can assume wlog that we are given  $F, G \in \mathcal{F}$  with  $S(i, m(i, T))(1, 1)(F) \neq F$  but  $S(i, m(i, T))(1, 1)(G) = G$ . Then  $(1, 1) \notin F$ .

Now if  $(1, 1) \in G$  then the intersection (of the two set and the kernel  $T$ ) increases by one and decreases by at most one, we are done. Otherwise  $S(i, m(i, T))(1, 1)(G) \in \mathcal{F}$ , and then we have  $t \leq |S(i, m(i, T))(1, 1)(G) \cap F \cap T| \leq |S(i, m(i, T))(1, 1)(F) \cap G \cap T|$ .



□

We are done with the proof of (i) of Lemma 2.6.

• proof of (ii):

Choose  $F, G \in \mathcal{F}$ :

*Case 1:* if

$$S(i, m(i, T))(n, 1)[\dots[S(i, m(i, T))(2, 1)[S(i, m(i, T))(1, 1)(F)]]\dots] = F' \neq F \text{ or}$$

$$S(i, m(i, T))(n, 1)[\dots[S(i, m(i, T))(2, 1)[S(i, m(i, T))(1, 1)(G)]]\dots] = G' \neq G,$$

since  $T$  is a  $t$ -kernel, however  $(i, m(i, T)) \notin F'$ , we are done similarly as in the previous claim.

*Case 2:* if

$$S(i, m(i, T))(n, 1)[\dots[S(i, m(i, T))(2, 1)[S(i, m(i, T))(1, 1)(F)]]\dots] = F \text{ and}$$

$$S(i, m(i, T))(n, 1)[\dots[S(i, m(i, T))(2, 1)[S(i, m(i, T))(1, 1)(G)]]\dots] = G,$$

then

a) if  $(i, m(i, T)) \notin F \cap G$ , we are done,

b) if  $(i, m(i, T)) \in F \cap G$  then since  $2 \leq m(i, T)$  and  $2k - t \leq n$ , there is  $j \leq n$  with  $(1, j) \notin F \cup G$ .

Then as we are in the *Case 2*,  $S(i, m(i, T))(j, 1)(F) \in \mathcal{F}$ , so

$$t \leq |S(i, m(i, T))(j, 1)(F) \cap G \cap T| = |F \cap G \cap (T \setminus (i, m(i, T)))|.$$

We are done with the proof of Lemma 2.6.

□

We are done with the proof of Theorem 1.6.

□

### 3 Proof of Theorem 1.7

Let  $\mathcal{G}_s = \{F \cap M(n, 1) : F \in f(\mathcal{F}), |F \cap M(n, 1)| = s\}$ . Let us consider  $G \in \mathcal{G}_s$  and examine the number of multisets  $F \in \mathcal{F}$  with  $G = F \cap M(n, 1)$ . Obviously  $k - s$  further elements belong to  $F$ , and they are in the same  $s$  columns, they can be chosen at most  $\binom{s + (k - s - 1)}{k - s}$  ways. Then we know that

$$|\mathcal{F}| = |f(\mathcal{F})| \leq \sum_{s=t}^k |\mathcal{G}_s| \binom{s + (k - s - 1)}{k - s} = \sum_{s=t}^k |\mathcal{G}_s| \binom{k - 1}{k - s}.$$

Now consider a family  $\mathcal{F}'$  of sets on an underlying set of size  $n+k-1$ . Let it be the same on the first  $n$ -elements as  $f(\mathcal{F})$  in  $M(n, 1)$ , and extend every  $s$ -element set there with all the  $k-s$ -element subsets of the remaining  $k-1$  elements of the underlying set. It can happen  $\binom{k-1}{k-s}$  ways, thus the cardinality of this family is the right hand side of the above inequality.

It is easy to see that  $\mathcal{F}'$  is  $t$ -intersecting, hence its cardinality is at most  $AK(n+k-1, k, t)$ , which finishes the proof of Theorem 1.7.

## 4 Concluding remarks

Note that for a family  $\mathcal{A}_{n+k-1, k, t, i}$  we can define a  $t$ -intersecting family of  $k$ -multisets in  $M(n, k)$ , hence the bound given in Theorem 1.7 is sharp. However, we do not know any nontrivial bounds in case  $n < 2k - t$ .

Another interesting problem is the case  $l < k$ . Theorem 1.6 gives us a small  $t$ -kernel, but the proof of Theorem 1.7 does not work in this case.

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